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Reliability-Based Design Optimization With Confidence Level for Non-Gaussian Distributions Using Bootstrap Method

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1 Introduction

Obtaining an accurate input statistical model, which includes marginal distributions and a joint distribution of input random variables, is crucial to obtain an accurate reliability-based optimum design. However, for many input random variables, such as loading, material properties, and manufacturing geometric variability, only limited data are available due to expensive testing costs. If an input statistical model is obtained from insufficient data, it could yield an unreliable optimum design. To deal with the input statistical uncertainties, the possibility-based design optimization (PBDO) [1,2], the interval-based method [3–5], and the Bayesian reliability-based design optimization (RBDO) have been recently proposed [6,7]. However, the PBDO and interval-based method do not consider data size to properly quantify the uncertainties, and the Bayesian RBDO method assumes that input distributions are known, which is not common in real applications. In this paper, to offset inaccurate estimation of the input model, generating an input model with a confidence level has been recently proposed by using adjusted standard deviations and a correlation coefficient [8].

The adjusted standard deviation and correlation coefficient proposed in Ref. 8 are obtained from the confidence intervals for the input distribution parameters such as the mean, standard deviation, and correlation coefficient. The confidence intervals for the mean and standard deviation are usually calculated using Gaussian distributions of the input variables [9]. If the input variables have marginal Gaussian distributions, the confidence intervals for the mean and standard deviation can be explicitly and exactly calculated [8].

However, if the input variables have non-Gaussian marginal distributions, the estimated confidence intervals obtained using the assumption of Gaussian distributions of the input variables will be inaccurate. Thus, a bootstrap method [10–12], which does not require an assumption of Gaussian distribution of the input variables, is used for the non-Gaussian distributions. Since the confidence interval for the standard deviation is inaccurately estimated for non-Gaussian distribution comparing with the one for the mean, a method of using fourth moment and empirically adjusted parameters for equalizing tail probabilities [13] or the modified percentile-*t* bootstrap method [14] can be used. However, the use of fourth moment may yield inaccurate or too conservative estimation of the confidence interval for small number of samples. In Refs. 13 and 14, the performance of the modified percentile-*t* bootstrap method was compared with only percentile-*t* and other modified percentile-*t* bootstrap methods, but not compared with various bootstrap methods. In this paper, the most representative bootstrap methods such as the normal approximation, percentile, bias corrected (BC), bias corrected and accelerated (BCa), and percentile-*t* methods are tested to find which method can estimate the confidence intervals for the standard deviation, the most accurately and provide appropriate length of the confidence interval to obtain desirable input and output confidence levels.

To validate whether use of the adjusted standard deviations and correlation coefficient obtained using the bootstrap method provides a desirable confidence level of the input model, the confidence level of the input model with adjusted parameters is assessed through simulation tests. The β_t–contour, for the given target reliability index β_t, is used to measure the confidence level of the input model. This is because for a gradient-based or the most probable point (MPP)-based RBDO, the MPP is located on the β_t–contour (see Sec. 2.1 for definition), and thus, the size of β_t–contour, which can be controlled by the standard deviations of the input random variables, determines how reliable the optimum

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design will be. A mathematical example and an M1A1 Abrams tank roadarm problem with non-Gaussian correlated variables are used to illustrate how the input model with a target confidence level using the bootstrap method provides a more desirable output confidence level compared with the one without using the proposed method for correlated input distributions. The proposed method is focused on correlated random variables only. Uncorrelated but dependent variables are beyond the scope of the paper.

2 Confidence Level of Input Model

When an input model is estimated from given limited data, we want to know how reliable the RBDO result obtained from the estimated input model is. Even though the target probability of failure is 2.275%, the probability evaluated at the RBDO optimum using the estimated input model could be much larger than the target probability of failure due to inaccurate estimation of the input model. However, it is difficult to predict an accurate confidence level of the output performance of the RBDO optimum because the confidence level of the output performance depends on problems even though the same input model is used. Thus, the confidence level of the input model needs to be first estimated before stepping into estimation of the confidence level of the output performance. Even though the confidence level of the input model is not necessarily equivalent to the output confidence level, if a conservative measure for estimating the input confidence level, i.e., a β_t–contour is used, then it can be assured that the confidence level of the output performance is at least larger than the confidence level of the input model. In this paper, the confidence level of the input model is defined as the probability that the β_t–contour obtained using the estimated input model covers the β_t–contour obtained using the true input model. Since a larger β_t–contour assures the RBDO optimum design to satisfy the target reliability, the confidence level of the input model will provide a confidence level that the RBDO optimum design satisfies the target reliability. This measure of input confidence level, β_t–contour, is explained in Sec. 2.1 in detail.

In Sec. 2.2, the adjusted parameters obtained from the confidence intervals for the input parameters are introduced.

2.1 Measure of Input Confidence Level.

The RBDO formulation is defined to

$$\begin{aligned}
&\text{min.}\quad \text{Cost}(\mathbf{d}) \\
&\text{s.t.}\quad P(G_j(\mathbf{X}) > 0) \le P_{F_j}^{\text{Tar}}, \quad j = 1, \ldots, nc \\
&\qquad \mathbf{d} = \boldsymbol{\mu}(\mathbf{X}), \quad \mathbf{d}_\mathbf{L} \le \mathbf{d} \le \mathbf{d}_\mathbf{U}, \quad \mathbf{d} \in R^{ndv} \text{ and } \mathbf{X} \in R^n
\end{aligned} \tag{1}$$

where $\mathbf{X}$ is the vector of random variables; $\mathbf{d}$ is the vector of design variables, which is the mean values of random variables; $G_j(\mathbf{X})$ represents the jth constraint function; $P_{F_j}^{\text{Tar}}$ is the given target probability of failure for the jth constraint; and nc, ndv, and n are the number of probabilistic constraints, design variables, and random variables, respectively. To satisfy the probabilistic constraint in Eq. (1), the constraint function evaluated at the MPP ($\mathbf{x}^*$), $G_j(\mathbf{x}^*)$ should be less than 0, where the MPP is obtained by solving the inverse reliability analysis as

$$\begin{aligned}
&\text{maximize}\quad g_j(\mathbf{u}) \\
&\text{subject to}\quad \|\mathbf{u}\| = \beta_{t_j}
\end{aligned} \tag{2}$$

where $g_j(\mathbf{u})$ is the jth constraint function in the standard normal $\mathbf{U}$–space, i.e., $g_j(\mathbf{u}) \equiv G_j(\mathbf{x}(\mathbf{u})) = G_j(\mathbf{x})$ and β_{t_j} is the target reliability index such that $P_{F_j}^{\text{Tar}} = \Phi\left(-\beta_{t_j}\right)$ for the jth constraint. Once the hyper-sphere in Eq. (2), $\|\mathbf{u}\| = \beta_{t_j}$, is transformed from the $\mathbf{U}$–space to the $\mathbf{X}$–space using the Rosenblatt transformation [15], it is called as β_t–contour, and the MPP search is carried out on the β_t–contour for the inverse reliability analysis in Eq. (2).

The Rosenblatt transformation uses a conditional joint distribution of input random variables, which is obtained from a joint distribution. However, since it is difficult to obtain a joint distribution directly from limited data, a copula function, which consists of marginal distributions and correlation parameters, is used to model the joint distribution as [16]

$$F_{X_1,\cdots,X_n}(x_1, \ldots, x_n) = C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)|\theta) \tag{3}$$

where $F_{X_1\cdots X_n}(x_1, \ldots, x_n)$ is the joint cumulative distribution function (CDF) of $X_1, \ldots, X_n$, and $F_{X_i}(x_i)$ is the marginal CDF of input random variables X_i for $i = 1, \ldots, n$. C is a copula function of the marginal CDFs with the given matrix of correlation parameters θ between $X_1, \ldots, X_n$. Since the correlation parameters are different for different copula types, a common correlation measure, the Kendall's tau [17] is used. The sample version of the Kendall's tau is calculated as

$$t = \frac{c-d}{c+d} = \frac{2(c-d)}{ns(ns-1)} \tag{4}$$

where c and d are the numbers of concordant and discordant pairs, respectively, and $c + d = \binom{ns}{2}$ where ns is the number of samples. A pair of two variable data sets (X_1,Y_1) and (X_2,Y_2) is called concordant if $(X_1-X_2)(Y_1-Y_2)>0$, and called discordant if $(X_1-X_2)(Y_1-Y_2)<0$. Once the Kendall's tau is obtained from samples using Eq. (4), the correlation parameter θ of two random variables can be implicitly obtained using the following equation:

$$\tau = 4\int\int_{I^2} C(u, v|\theta)dC(u, v|\theta) - 1 \tag{5}$$

where $u = F_{X_1}(x_1)$ and $v = F_{X_2}(x_2)$ are the marginal CDFs of X_1 and X_2, respectively, $dC(u, v|\theta) = \frac{\partial^2 C(u,v|\theta)}{\partial u \partial v} dudv$, and the unit square I^2 is the product $I \times I = [0, 1] \times [0, 1]$ of the domain of the two marginal CDFs u and v. For some copulas, there exist explicit formulas between θ and τ, which are given in Ref. [18].

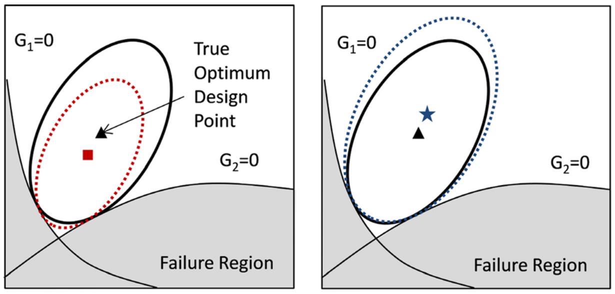
Using the copula function, the Rosenblatt transformation can be written as

$$\begin{aligned}
u_i &= \Phi^{-1}[F_{X_i}(x_i|x_1, x_2, \ldots, x_{i-1})] \\
&= \Phi^{-1}\left[\frac{\partial^{i-1}C(z_1, \ldots, z_{i-1})}{\partial z_1 \ldots \partial z_{i-1}} \cdot \frac{f_{X_1}(x_1)\ldots f_{X_{i-1}}(x_{i-1})}{f_{X_1\cdots X_{i-1}}(x_1, \ldots, x_{i-1})}\right]
\end{aligned} \tag{6}$$

where $F_{X_i}(x_i|x_1, x_2, \ldots, x_{i-1})$ is the conditional distribution function of X_i given $X_1, \ldots, X_{i-1}$; $z_i = F_{X_i}(x_i)$, and $f_{X_i}(x_i)$ and $f_{X_1\cdots X_{i-1}}(x_1, \ldots, x_{i-1})$ are the marginal and joint probability density function (PDF) for $i = 1,\ldots,n$. Inserting Eq. (6) into the second equation of Eq. (2), the hyper-sphere in the $\mathbf{U}$–space can be expressed as the β_t–contour. Most copula applications consider bivariate data because only few copula families have n-dimensional generalization. In engineering applications, such as the strain-based fatigue analysis, the fatigue strength coefficient and exponent are correlated and the fatigue ductility coefficient and exponent are correlated. However, they are known not to be cross correlated [19,20]. There are other types of problems such that the random variables are pair-wise correlated [21,22]. Thus, in this paper, only the bivariate copulas are considered.

The β_t–contour acts as a safety barrier that locates the optimum design point away from the constraint boundary with the target probability of failure. Therefore, if the β_t–contour is large enough for the optimum design point to be away from the constraint boundary with the target probability of failure, it means that the obtained optimum design satisfies the target reliability.

Consider two β_t–contours obtained from an estimated input model (dotted line) and the true input model (solid line) shown in



(a) Smaller β_t −contour (b) Larger β_t −contour

Fig. 1 β_t–contours using true and estimated input model

Figs. 1(*a*) and 1(*b*). According to the estimated input models, the β_t–contours can be smaller or larger than the true β_t–contour shown in Figs. 1(*a*) and 1(*b*), respectively. In Fig. 1(*a*), since the smaller β_t–contour yields the optimum design point (rectangle) closer to the constraint boundaries than the true optimum design (triangle), the probabilities of failure of the constraints evaluated at the obtained optimum design will be larger than the target probability of failure. On the other hand, in Fig. 1(*b*), the probabilities of failure of the constraints evaluated at the optimum design using the larger β_t–contour (star) will be smaller than the target probability of failure. Accordingly, the β_t–contour using the estimated input model needs to be at least larger than the β_t–contour using the true input model for the optimum design to satisfy the target reliability.

2.2 Adjusted Parameters for Obtaining Input Model With Confidence Level. The input parameters of the marginal distribution and copula, such as the mean, standard deviation, and correlation coefficient, determine the shape and size of the β_t–contour as shown in Eqs. (2) and (6). The estimated input parameters, such as the sample mean, standard deviations and correlation coefficients, include prediction errors. Thus, the confidence intervals for these parameters, which is likely to include unknown population parameters with a confidence level, e.g., 95%, need to be considered to obtain the input model that provides an optimum design to satisfy the target reliability with the same confidence level. Since the confidence level indicates with what probability the population parameter falls within the confidence interval, the larger the confidence level is, the bigger the confidence interval for the estimated parameter is.

Among these parameters, the standard deviation determines the size of the β_t–contour; hence, the upper bound of confidence interval for the estimated standard deviation yields larger β_t–contour such as Fig. 1(*b*), which assures the optimum design to satisfy the target reliability. On the other hand, the mean determines the location of the β_t–contour, and the correlation coefficient determines the shape of the β_t–contour. Accordingly, it is not clear that the upper or lower bound of the confidence intervals for the estimated mean and correlation coefficient yields a reliable design. Instead of directly using the lower or upper bound of the confidence interval for the input parameters, adjusted parameters are introduced to cover the prediction error of all input parameters.

Let the change in the sample standard deviation caused by the change in the sample mean $\Delta\tilde{\mu} = |\tilde{\mu}^U - \tilde{\mu}| = |\tilde{\mu} - \tilde{\mu}^L|$ be $\Delta\tilde{\sigma}$. Since the coefficient of variation (COV), $\tilde{\sigma}/\tilde{\mu}$, is proportional to the prediction error of the mean, $\Delta\tilde{\sigma}$ needs to be enlarged in proportion to the change in the sample mean. Thus, it is proposed that the ratio of $\Delta\tilde{\sigma}$ to $\Delta\tilde{\mu}$ is approximated as the COV, $\Delta\tilde{\sigma}/\Delta\tilde{\mu} \cong \tilde{\sigma}/\tilde{\mu}$. Adding $\Delta\tilde{\sigma}$ to the upper bound $\tilde{\sigma}^U$ of the confidence interval for the standard deviation, the adjusted standard deviation is heuristically obtained as

$$\tilde{\sigma}^A = \tilde{\sigma}^U + \Delta\tilde{\sigma} \cong \tilde{\sigma}^U + \frac{\tilde{\sigma}}{\tilde{\mu}} \times \Delta\tilde{\mu} \tag{7}$$

Notice that Eq. (7) should be used for the random parameters that do not change during design optimization. For the random variables, $\Delta\tilde{\sigma}$ is removed in Eq. (7) because the mean values of the random variables are design variables and thus controllable during the design optimization.

As explained before, either the lower or upper bound of the confidence interval for the correlation coefficient does not yield a reliable design. Instead, the small correlation coefficient combined with the adjusted standard deviation yields a reliable design than the large correlation coefficient with the adjusted standard deviation. Accordingly, the adjusted correlation coefficient is selected such that the ratio of $\tilde{\tau} - \tilde{\tau}^A$ to $\max(|\tilde{\tau}^U - \tilde{\tau}|, |\tilde{\tau} - \tilde{\tau}^L|)$ is equivalent to the estimated correlation coefficient $\tilde{\tau}$ as

$$\tilde{\tau}^A = \tilde{\tau} - \tilde{\tau} \times \max\left(\left|\tilde{\tau}^U - \tilde{\tau}\right|, \left|\tilde{\tau} - \tilde{\tau}^L\right|\right) \tag{8}$$

In Eq. (8), for the small correlation coefficient, the adjusted correlation coefficient is closer to the estimated correlation coefficient, and for the large correlation coefficient, it is closer to the lower bound of the confidence interval for the correlation coefficient. As the number of samples increases, the adjusted correlation coefficient converges to the true correlation coefficient.

For many marginal distributions, the input parameters can be expressed in terms of the mean and standard deviation. Since the mean and standard deviation determine the location and variability of the distributions regardless of distribution types, the standard deviations for non-Gaussian distributions can be used to enlarge the β_t–contour for RBDO. Accordingly, once the mean and standard deviation are calculated from the given data, the input parameters can be calculated using the explicit functions, which are presented for various marginal distributions in Ref. 18.

The adjusted parameters in Eqs. (7) and (8) for Gaussian input random variables are calculated explicitly and exactly [8]. However, if the input random variables are not Gaussian, the assumption that input random variables have Gaussian distribution may yield inaccurate confidence intervals, which lead to unreliable optimum designs. Thus, the bootstrap method is proposed to calculate the confidence intervals for the distribution parameters for non-Gaussian distributions in this paper.

3 Estimation of Input Statistical Model With Confidence Level

For obtaining RBDO result, the input model, i.e., the marginal distributions and joint distribution, needs to be estimated from the given data. In Secs. 3.1 and 3.2, identification and quantification of the marginal and joint distributions using bootstrap methods are explained. Since accurate estimation of the upper bound of the standard deviation is important to have a desirable input confidence level, various bootstrap methods for accurate estimation of the confidence interval for the standard deviation are discussed and tested using simulation studies in Sec. 3.3. In Sec. 3.4, a description of how the confidence interval for the correlation coefficient is calculated is provided. Then, the confidence level of the input model generated using the adjusted parameters, which are obtained from confidence intervals for input parameters in Eqs. (7) and (8), will be tested for various input distributions in Sec. 3.5.

3.1 Identification of Input Model. To identify the joint or marginal distribution types from the given data, a Bayesian method is used to select one candidate distribution that best describes the given data among candidate marginal or joint distributions. The input model can be identified by a one-step procedure, which directly tests all candidate marginal and joint distributions simultaneously, or by a two-step procedure, which first identifies marginal distributions and then a copula [18,23–26]. The two-step procedure is more efficient and accurate than the one-step procedure [27]. For example, if seven candidate

marginal distributions of X_1 and X_2 and nine candidate copulas are used to identify a joint distribution, the one-step procedure requires to test $7 \times 7 \times 9 = 441$ cases, whereas the two-step procedure requires to test $7 + 7 + 9 = 23$ cases. It is more challenging to identify a correct joint distribution from 441 candidates compared to 23 candidates, so the two-step procedure is preferred.

Based on the measure of identification, the weight-based method [18] and Markov chain Monte Carlo (MCMC)-based method [28] can be used. The weight-based method calculates the weights of all candidates by integrating the likelihood functions of the candidates, and then selects one candidate with the highest weight. The MCMC-based method identifies a correct distribution among candidates using a criterion such as a deviance information criterion. However, the MCMC-based method uses random samples of the posterior distribution, which causes randomness of identification results. Thus, in this paper, a two-step weight-based Bayesian method is used to identify the input model. More detailed information on the two-step weight-based method is presented in Ref. 18.

3.2 Quantification of Input Model Using Bootstrap Method. If the input random variables have a Gaussian distribution, the confidence intervals for the mean and standard deviation can be explicitly and exactly obtained. However, if not, there are no explicit functions for calculation of the confidence intervals for the mean and standard deviation for a non-Gaussian distribution, so a bootstrap method needs to be introduced to obtain accurate confidence intervals for the mean and standard deviation for non-Gaussian distributions.

Because the confidence interval for the mean obtained assuming input variables have a Gaussian distribution can be accurately estimated even for most non-Gaussian distributions [29,30], the bootstrap method is tested to calculate the confidence interval for the standard deviation only. However, for non-Gaussian distributions with extremely high skewness such as highly skewed extreme distribution, the bootstrap method needs to be used to calculate the confidence interval for the mean as well as the standard deviation.

If the input variable follow a Gaussian distribution with the mean μ and standard deviation σ, the lower and upper bounds of the confidence interval for the mean ($\tilde{\mu}^L$ and $\tilde{\mu}^U$) can be obtained as [8,9]

$$\tilde{\mu}^L = \tilde{\mu} - t_{\alpha/2,ns-1}\frac{\tilde{\sigma}}{\sqrt{ns}} \quad \text{and} \quad \tilde{\mu}^U = \tilde{\mu} + t_{\alpha/2,ns-1}\frac{\tilde{\sigma}}{\sqrt{ns}} \tag{9}$$

where ns is the number of samples, $\tilde{\mu}$ and $\tilde{\sigma}$ are the sample mean and sample standard deviation, respectively, and $t_{\alpha/2,ns-1}$ is the value of the student's t-distribution with ($ns-1$) degrees of freedom at two-sided confidence level, $100 \times (1-\alpha/2)$. α indicates a significance level, which is the probability that the true mean is not within the confidence interval in Eq. (9). For example, for the given 95% confidence level, $\alpha = 0.05$.

Using a similar procedure of calculating the confidence interval for the mean, the lower and upper bounds of the confidence interval for the standard deviation of Gaussian distribution, $\tilde{\sigma}^L$ and $\tilde{\sigma}^U$, respectively, are calculated as [8,9]

$$\tilde{\sigma}^L = \sqrt{\frac{(ns-1)\tilde{\sigma}^2}{c_{1-\alpha/2,ns-1}}} \quad \text{and} \quad \tilde{\sigma}^U = \sqrt{\frac{(ns-1)\tilde{\sigma}^2}{c_{\alpha/2,ns-1}}} \tag{10}$$

where $c_{\alpha/2,ns-1}$ and $c_{1-\alpha/2,ns-1}$ are the critical values of the chi-square distribution evaluated at two-sided confidence level $100 \times (\alpha/2)$ and $100 \times (1-\alpha/2)$ with ($ns-1$) degrees of freedom, respectively.

For non-Gaussian distributions, the assumption that input variables have Gaussian distributions may yield inaccurate estimation of the confidence interval for the standard deviation. Thus, a bootstrap method, which does not require any assumption on the distribution types of input variables, needs to be used. To calculate the confidence interval for an estimated standard deviation $\tilde{\sigma}$, the bootstrap method constructs a distribution of the standard deviation using the frequency distribution of $\tilde{\sigma}^*$ obtained from randomly generated bootstrap samples based on the given data.

The first step is to construct an empirical distribution $F_{ns}(x)$ or a parametric distribution $\tilde{F}(x)$ from the given samples, $\mathbf{x} = [x_1, x_2, \ldots, x_{ns}]$. If a random sample of size ns with replacement is drawn from the empirical distribution $F_{ns}(x)$, then this is called a nonparametric approach. If the resample is drawn from the specified model $\tilde{F}(x)$ determined from the given samples, this is called a parametric approach. The empirical distribution is obtained as

$$F_{ns}(x) = \frac{1}{ns}\sum_{i=1}^{ns} I\{x_i \le x\} \tag{11}$$

where $I\{\mathrm{A}\}$ is the indicator function of event A, that is, $I\{\mathrm{A}\} = 1$ if the event A occurs, otherwise, $I\{\mathrm{A}\} = 0$.

In the second step, bootstrap samples are generated from an empirical distribution or parametric distribution. The third step is to calculate $\tilde{\sigma}$ from the resample, drawn from either an empirical or parametric distribution, yielding $\tilde{\sigma}^*_{bs}$. In the fourth step, the second and third steps are repeated B times (e.g., $\mathrm{B} = 1000$). Then, the fifth step is to construct a probability distribution from $\tilde{\sigma}^*_1, \tilde{\sigma}^*_2, \ldots, \tilde{\sigma}^*_B$. This distribution is the bootstrap sampling distribution $\tilde{G}^*(\tilde{\sigma}^*)$ of $\tilde{\sigma}$, which is used to calculate the confidence interval for $\tilde{\sigma}$. To obtain the bootstrap sampling distribution of $\tilde{\sigma}$, five bootstrap methods such as the normal approximation, percentile, bias corrected, percentile-t, or bias corrected accelerated methods can be used and their performances are tested in this paper. Table 1 summarizes how to calculate the confidence interval for the standard deviation using the bootstrap method.

3.2.1 Normal Approximation Method. The normal approximation method assumes that the distribution of estimated standard deviation is a Gaussian distribution. Using the assumption, the confidence interval for the standard deviation is obtained as [8,31]

$$\tilde{\sigma} - z_{\alpha/2}\tilde{\sigma}^*_{\tilde{\sigma}} < \sigma < \tilde{\sigma} + z_{\alpha/2}\tilde{\sigma}^*_{\tilde{\sigma}} \tag{12}$$

where $\tilde{\sigma}^*_{\tilde{\sigma}} = \sqrt{\sum_{bs=1}^{B}\left[\tilde{\sigma}^*_{bs} - \tilde{\sigma}^*\right]^2/(B-1)}$, $\tilde{\sigma}^* = \sum_{bs=1}^{B}\tilde{\sigma}^*_{bs}/B$, and $z_{\alpha/2}$ are the values of standard Gaussian distribution CDF at $\alpha/2$.

3.2.2 Percentile Method. The percentile method calculates the confidence interval for the parameter based on the bootstrap sampling distribution $\tilde{G}^*(\tilde{\sigma}^*)$ approximating the population distribution $G(\tilde{\sigma})$. The basic idea of this method is that the confidence interval for $(1-\alpha)$ level includes all the values of $\tilde{\sigma}^*$ between the $(\alpha/2 \times 100)$th and $(1-\alpha/2) \times 100$th percentiles of $\tilde{G}^*(\tilde{\sigma}^*)$. The sorting vector of $\tilde{\sigma}^*_{bs}$ is obtained from each bootstrap sample for $bs = 1, \ldots, B$ and the values of $\tilde{\sigma}^*_{bs}$ evaluated at the $(\alpha/2 \times 100)$th

Table 1 Bootstrap procedure

	Bootstrap procedures
Step 1	From given samples $\mathbf{x} = [x_1, x_2, \ldots, x_{ns}]$, construct an empirical distribution $F_{ns}(x)$ or parametric distribution $\tilde{F}(x)$.
Step 2	Generate bootstrap samples $\mathbf{x}^* = [x_1^*, x_2^*, \ldots, x_{ns}^*]$ from the constructed distribution in Step 1.
Step 3	Calculate a statistic of interest $\tilde{\sigma}$ from bootstrap samples, yielding $\tilde{\sigma}^*_{bs}$, $bs = 1, \ldots, B$
Step 4	Repeat Step 2 and 3 B times (e.g., $\mathrm{B} = 1000$).
Step 5	Construct a probability distribution $\tilde{G}^*(\tilde{\sigma}^*)$ from $\tilde{\sigma}^*_1, \tilde{\sigma}^*_2, \ldots, \tilde{\sigma}^*_B$, and then calculate confidence interval for estimated parameter, $\tilde{\sigma}$ using $\tilde{G}^*(\tilde{\sigma}^*)$.

and $(1-\alpha/2)\times 100$th percentiles of $\tilde{G}^*(\tilde{\sigma}^*)$ are used as the lower and upper bounds of $\tilde{\sigma}$, respectively,

$$\tilde{\sigma}^*_{\alpha/2} < \sigma < \tilde{\sigma}^*_{1-\alpha/2} \tag{13}$$

Since the percentile method does not assume that the bootstrap sampling distribution follows a Gaussian distribution such as the normal approximation method, it allows $\tilde{G}^*(\tilde{\sigma}^*)$ conforming to any shape that the data follow. For this reason, it is the most widely used bootstrap technique among applied statisticians [32]. However, when the number of samples is small, $\tilde{G}^*(\tilde{\sigma}^*)$ might be a biased estimator of $G(\tilde{\sigma})$, i.e., $\tilde{\sigma}^*$ is a biased estimator of $\tilde{\sigma}$. In that case, the percentile method can be inaccurate.

3.2.3 BC Method. The BC method corrects the bias term by introducing an adjusted parameter z_0. Suppose that there exist some monotonic transformations between $\tilde{\sigma}^*$ and $\tilde{\sigma}$, say, $\tilde{\varphi}-\varphi$. Instead of assuming that $\tilde{\sigma}^*-\tilde{\sigma}$ is centered on zero, the BC method assumes that $\varphi(\tilde{\sigma}^*)-\varphi(\tilde{\sigma})+z_0=z$ follows a standard Gaussian distribution. Since $\tilde{\varphi}$ and φ are monotonic functions, it holds that $\Pr(\tilde{\sigma}^*\le\tilde{\sigma})=\Pr(z\le z_0)=\Phi(z_0)$, where $\Phi(\cdot)$ is the standard Gaussian CDF. Accordingly, z_0 is calculated using [31]

$$z_0=\Phi^{-1}(\Pr(\tilde{\sigma}^*\le\tilde{\sigma})) \tag{14}$$

where z_0 is a biasing constant that compensates the bias between $\tilde{\sigma}^*$ and $\tilde{\sigma}$, and $\Pr(\tilde{\sigma}^*\le\tilde{\sigma})=\frac{1}{B}\sum_{i=1}^{bs} I\left\{\tilde{\sigma}^*_{bs}\le\tilde{\sigma}\right\}$. Since $\tilde{G}^*(\tilde{\sigma}^*)$ is invariant to the transformation, the transformation does not need to be known. Using z_0, the confidence interval for $\tilde{\sigma}$ is obtained as

$$\tilde{\sigma}^*_{\Phi\left(2z_0+z_{\alpha/2}\right)} < \sigma < \tilde{\sigma}^*_{\Phi\left(2z_0+z_{1-\alpha/2}\right)} \tag{15}$$

where $\tilde{\sigma}^*_{\Phi\left(2z_0+z_{\alpha/2}\right)}$ is the value of $\tilde{\sigma}^*$ evaluated at the $\Phi\left(2z_0+z_{\alpha/2}\right)\times 100$th percentile and $\tilde{\sigma}^*_{\Phi\left(2z_0+z_{1-\alpha/2}\right)}$ is the value of $\tilde{\sigma}^*$ evaluated at the $\Phi\left(2z_0+z_{1-\alpha/2}\right)\times 100$th percentile. The BC method corrects the bias term, but it still requires the parametric assumption that there exist monotonic transformations between $\tilde{\sigma}^*$ and $\tilde{\sigma}$.

3.2.4 BCa Method. The BCa method generalizes the BC method in a way that the BC method only corrects the bias, whereas the BCa method corrects both the bias and the skewness. The BCa method assumes that, for certain monotonic transformations $\tilde{\varphi}$ and φ, the bias constant z_0 and acceleration constant A result in Ref. [11]

$$\frac{(\tilde{\varphi}-\varphi)}{\sigma_{\tilde{\varphi}}}\sim N\left(-z_0\sigma_h,\sigma_h^2\right),\sigma_h=1+Ah \tag{16}$$

where $\sigma_{\tilde{\varphi}}$ is the constant standard error of $\tilde{\varphi}$. The acceleration A is defined as

$$A=\frac{\sum_{i=1}^{ns}\left(\tilde{\sigma}-\tilde{\sigma}_{(i)}\right)^3}{6\left\{\sum_{i=1}^{ns}\left(\tilde{\sigma}-\tilde{\sigma}_{(i)}\right)^2\right\}^{3/2}} \tag{17}$$

where $\tilde{\sigma}_{(i)}$ is the estimated parameter of $\mathbf{x}_{(i)}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{ns})$ without the ith point x_i and $\tilde{\sigma}=\sum_{i=1}^{ns}\tilde{\sigma}_{(i)}/ns$. Using Eq. (16), the BCa confidence interval is defined as

$$\tilde{\sigma}^*_{\alpha_1}<\sigma<\tilde{\sigma}^*_{\alpha_2} \tag{18}$$

where $\alpha_1=\Phi\left(z_0+\frac{z_0+z_{\alpha/2}}{1-A\left(z_0+z_{\alpha/2}\right)}\right)$, $z_0=\Phi^{-1}(\Pr(\tilde{\sigma}*\le\tilde{\sigma}))$, and $\alpha_2=\Phi\left(z_0+\frac{z_0+z_{1-\alpha/2}}{1-A\left(z_0+z_{1-\alpha/2}\right)}\right)$.

However, since the BCa method highly depends on the acceleration A, if it is not accurate, the BCa is also inaccurate.

3.2.5 Percentile-t Method. The percentile-*t* method uses the distribution of a standardized estimator to calculate the confidence interval. The percentile-*t* interval is expected to be accurate to the extent that standardizing depends less on the bootstrap sampling estimator, $\tilde{\sigma}^*$, than the percentile method. The standardized parameter t^*_{bs} can be defined as [32]

$$t^*_{bs}=\left(\tilde{\sigma}^*_{bs}-\tilde{\sigma}\right)/\tilde{\sigma}_{\tilde{\sigma}^*_{bs}} \tag{19}$$

where $\tilde{\sigma}^*_{bs}$ is the estimated parameter from each resampled data and $\mathbf{x}^*_{bs}=\left[x^*_{1bs},x^*_{2bs},\ldots,x^*_{nsbs}\right]$ for $bs=1,\ldots,B$. In Eq. (19), $\tilde{\sigma}$ is the estimated parameter from the original data, $\mathbf{x}=\left[x_1,x_2,\ldots,x_{ns}\right]$ and $\tilde{\sigma}_{\tilde{\sigma}^*_{bs}}$ is the standard deviation of $\tilde{\sigma}$ obtained from a double bootstrap, which is another level of resampling. That is, the double bootstrap sample $\mathbf{x}^*_d=\left[x^{**}_{1d},x^{**}_{2d},\ldots,x^{**}_{nsd}\right]$ for $d=1,\ldots,D$ is resampled from the bootstrap samples, $\mathbf{x}^*_{bs}=\left[x^*_{1bs},x^*_{2bs},\ldots,x^*_{nsbs}\right]$ for $bs=1,\ldots,B$. Thus, the percentile-*t* method requires a large number $(D\times B)$ of bootstrap samples. Using the double bootstrap samples, $\tilde{\sigma}_{\tilde{\sigma}^*_{bs}}$ is obtained as

$$\tilde{\sigma}_{\tilde{\sigma}^*_{bs}}=\sqrt{\frac{\sum_{d=1}^{D}\left[\tilde{\sigma}^{**}_d-\tilde{\sigma}^{**}\right]^2}{D-1}} \tag{20}$$

where $\tilde{\sigma}^{**}=\sum_{d=1}^{D}\tilde{\sigma}^{**}_d/D$.

The confidence interval for the sample standard deviation $\tilde{\sigma}$ is obtained as

$$\tilde{\sigma}+t^*_{\alpha/2}\tilde{\sigma}_{\tilde{\sigma}}<\sigma<\tilde{\sigma}+t^*_{1-\alpha/2}\tilde{\sigma}_{\tilde{\sigma}} \tag{21}$$

where $t^*_{\alpha/2}$ and $t^*_{1-\alpha/2}$ are the values of t^* evaluated at $\alpha/2\times 100$th and $(1-\alpha/2)\times 100$th percentiles. In Eq. (21), $\tilde{\sigma}_{\tilde{\sigma}}$ is the population standard deviation, which is calculated by $\tilde{\sigma}_{\tilde{\sigma}}=\sqrt{\sum_{d=1}^{D}\left[\tilde{\sigma}^*_d-\tilde{\sigma}^*\right]^2/(D-1)}$, where $\tilde{\sigma}^*_d$ is the estimator obtained from the resampled data, $\mathbf{x}^*=\left[x^*_1,x^*_2,\ldots,x^*_D\right]$, which are randomly generated from the original data.

3.3 Tests of Bootstrap Methods. To test and compare the performance of the five bootstrap methods, a lognormal distribution is considered as the true marginal distribution with $\mu=5.0$ and $\sigma=5.0$, which means a relatively large COV. For calculation of confidence interval for the standard deviation, an empirical distribution (nonparametric approach) or a parametric distribution (parametric approach) needs to be first constructed as explained in Sec. 3.2. For the parametric approach, a distribution type and its parameters need to be determined from samples, which are randomly generated from the assumed true marginal distribution (i.e., lognormal distribution). The distribution type is identified from the generated samples over 1000 data sets with different sample sizes of $ns=30$, 100, and 300 using the two-step weight-based Bayesian method where seven candidate distributions are the Gaussian, Weibull, gamma, lognormal, Gumbel, extreme, and extreme type II distributions.

The distribution type is differently identified for different data sets. In this testing, the true lognormal distribution is identified 650 times out of the 1000 data sets from the seven candidates for $ns=30$. Weibull and gamma distributions are identified 113 and 210 times, respectively, because they have the most close distribution shapes to the lognormal distribution with $\mu=5$ and $\sigma=5$ among the seven candidates. For $ns=100$ and 300, the correct distribution is identified 875 and 980 times out of the 1000 data

Table 2 Obtained confidence levels (%) of confidence interval for standard deviation

ns	Approach	Normal approximation	Percentile	BC	BCa	Percentile-t
30	Nonpar.	64.9	61.3	64.8	70.4	83.7
	Par. (Iden.)	74.9	79.0	85.0	88.9	89.6
	Par. (True)	89.3	95.2	88.3	92.4	88.3
100	Nonpar.	76.0	75.1	78.1	83.1	88.8
	Par. (Iden.)	87.2	87.5	87.4	89.9	92.0
	Par. (True)	95.2	97.7	86.9	89.9	87.1
300	Nonpar.	82.3	82.7	84.5	87.9	91.3
	Par. (Iden.)	95.9	97.5	86.8	88.4	85.7
	Par. (True)	96.7	99.0	86.3	88.2	85.3

Note: Method using Gaussian distribution of input variable yields 65.8% for $ns = 30$, 66.0% for $ns = 100$, and 65.0% for $ns = 300$.

Table 3 Mean values of upper bound of confidence interval for standard deviation

ns	Approach	Normal approximation	Percentile	BC	BCa	Percentile-t
30	Nonpar.	6.900	6.458	6.803	7.379	20.01
	Par. (Iden.)	7.830	8.396	12.72	15.22	16.62
	Par. (True)	8.742	9.998	12.57	15.83	16.04
100	Nonpar.	6.482	6.367	6.604	7.144	10.78
	Par. (Iden.)	7.167	7.596	9.204	10.43	10.83
	Par. (True)	7.367	7.978	9.342	10.60	10.77
300	Nonpar.	6.088	6.078	6.219	6.571	7.810
	Par. (Iden.)	6.416	6.763	7.324	7.877	7.642
	Par. (True)	6.491	6.824	8.034	7.422	7.984

Note: Method using Gaussian distribution of input variable yields 6.451 for $ns = 30$, 5.708 for $ns = 100$, and 5.401 for $ns = 300$.

sets, respectively. That is, as the number of samples increases, the identification error decreases.

Using the obtained parametric distribution, the bootstrap method calculates the upper bound of the confidence interval for the standard deviation. The confidence level of the standard deviation is assessed by calculating the probability that the upper bound of the confidence interval for the standard deviation obtained from each data set is larger than the true standard deviation.

To observe the effect of the identification error on calculation of the upper bound for the standard deviation, the confidence level using the identified distribution is compared in Table 2 with the one using the true distribution. For 95% of two-sided target confidence level, the target confidence level is 97.5% for the upper side of the confidence interval. The most desirable confidence interval would just include the true standard deviation with the target confidence level.

Table 2 shows the obtained confidence level using the five bootstrap methods. Since the distribution of the standard deviation is highly skewed due to the lognormal distribution with large COV, all methods have poor confidence levels, especially for $ns = 30$. As the number of samples increases, the obtained confidence levels are getting closer to the target confidence level, whereas the method assuming Gaussian distribution of input variable in Eq. (10) provides only around 65% as indicated at the bottom of Table 2 regardless of the number of samples. The parametric approach using the true marginal CDF has the best performance, and the parametric approach with identified CDF is not as good as the one with true CDF due to the identification. However, it is better than the nonparametric approach because it yields more desirable confidence level than the nonparametric approach when the identified CDF types are correct.

As shown in Table 2, when the nonparametric approach is used, the percentile-t has the highest confidence level, followed by the BCa method among five bootstrap methods. For the parametric approach using identified CDF, the percentile-t method is the best for $ns = 30$ and 100, and the percentile method is the best for $ns = 300$. When the parametric approach using the true CDF type is used, the percentile method has the best performance for $ns = 30$, 100, and 300.

Even though the percentile-t method provides more desirable confidence levels than other methods for the nonparametric approach, it provides unnecessarily large upper bounds of standard deviation. For example, the mean value of the upper bound of standard deviation using the nonparametric percentile-t method is 20.01 as shown in Table 3, which is significantly larger than the true standard deviation 5.0. In addition, it has a large standard deviation, 32.78, of the upper bounds of confidence intervals for standard deviation as shown in Table 4. That is, the upper bounds of confidence intervals for the standard deviation are overestimated and spread over the wide range of the standard deviation. As the number of samples increases, the upper bound of the standard deviation approaches the true standard deviation and its variation is reduced, but it converges very slowly to the true standard deviation compared to other methods. Thus, the BCa method, which has second highest confidence level and adequate value of upper bound, is preferred for the nonparametric approach.

Likewise, for the parametric approach, the percentile method yields a desirable confidence level and has adequate values of upper bounds for standard deviation and small variations as shown in Tables 3 and 4. Hence, the parametric percentile method is used in this paper for computing the confidence intervals for non-Gaussian random variables.

Figure 2 shows histograms of the upper bounds of the confidence interval for the standard deviation using the parametric percentile bootstrap method with identified CDF. The estimated upper bounds of the standard deviations mostly centered at the true standard deviation (5.0) even for $ns = 30$. As the number of samples increases, a large amount of the upper bounds of standard deviation tends to be very close to the true standard deviation with a small variation.

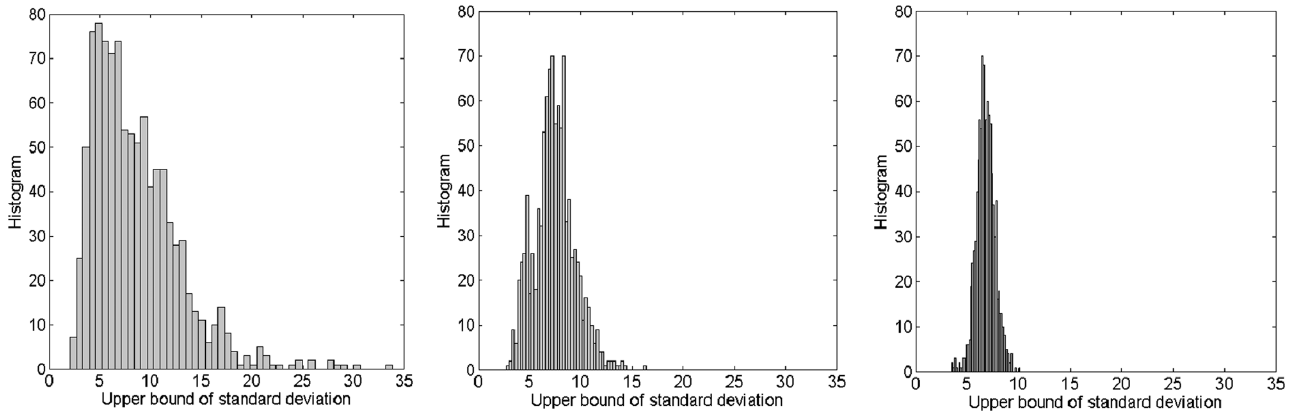
As shown in the example, for the lognormal distribution with a large COV such as 1.0, the performances of the BC, BCa, and percentile-t methods are not as good as the percentile method. This is because non-Gaussian distribution with a large COV has a large variation of bias corrected terms, which may yield an over conservative confidence interval. Likewise, the double bootstrap samples in percentile-t method depend more on the estimated distribution than other methods, but the estimated parameters are not accurately estimated for the non-Gaussian distribution with a large COV, which leads to inaccurate estimation of confidence interval for the standard deviation. Even though the bootstrap methods do not achieve the target confidence level for a small number of samples, as the number of samples increases, the obtained confidence levels tend to converge to the target confidence level while the method assuming Gaussian distribution of input variable does not.

When the input variable has a Gaussian distribution, the method using the Gaussian distribution of the input variable needs to be

Table 4 Standard deviations of upper bound of confidence interval for standard deviation

ns	Approach	Normal approximation	Percentile	BC	BCa	Percentile-t
30	Nonpar.	3.629	3.130	3.491	3.990	32.78
	Par. (Iden.)	3.873	4.316	11.62	12.96	20.66
	Par. (True)	3.587	3.958	10.18	11.81	21.21
100	Nonpar.	2.272	2.132	2.365	2.813	11.71
	Par. (Iden.)	1.977	2.019	4.725	5.373	12.34
	Par. (True)	1.917	1.663	4.790	5.520	12.77
300	Nonpar.	1.219	1.200	1.308	1.594	3.930
	Par. (Iden.)	0.971	0.873	2.728	2.978	4.692
	Par. (True)	0.929	0.815	2.598	2.946	4.695

Note: Method using Gaussian distribution of input variable yields 2.484 for $ns = 30$, 1.488 for $ns = 100$, and 0.834 for $ns = 300$.



(a) ns=30 (b) ns=100 (c) ns=300

Fig. 2 Histograms of upper bound of confidence interval for standard deviation

used because it has an exact formulation of calculating the confidence interval for the standard deviation. Even in that case, the bootstrap method can be used, but it might not be as accurate as the method using the Gaussian distribution of input variable because of the randomness of the bootstrap samples. The bootstrap method can be applied to any types of distribution, and the test results for various types of distributions are presented in Ref. 30.

The accuracy of the confidence interval for the standard deviation could be improved by including higher moments such as the skewness or fourth moment [12]. However, the higher moments could yield quite inaccurate confidence interval especially for small number of samples because accuracy of the estimated confidence interval highly depends on the accuracy of the estimated moments from given samples. Even though the parametric percentile method does not achieve the target confidence level for small number of samples as shown in Table 2, it generates an adequate length of confidence interval for the standard deviation and the obtained confidence level converges to the target confidence level as the number of samples increases.

3.4 Confidence Interval for Correlation Coefficient. It is known that as the number of samples ns goes to infinity, the sample correlation parameter follows a Gaussian distribution as [33]

$$\tilde{\theta} \sim N\left(\theta, \frac{1}{ns}\left\{4w\frac{dg^{-1}(\tilde{\tau})}{d\tilde{\tau}}\right\}^2\right) \quad (22)$$

where $\theta = g^{-1}(\tau)$, $w = \sqrt{\frac{1}{ns}\sum_{i=1}^{ns}(w_i + \tilde{w}_i - 2\bar{w})^2}$, $w_i = \frac{1}{ns}\sum_{j=1}^{ns} I_{ij}$, $\tilde{w}_i = \frac{1}{ns}\sum_{j=1}^{ns} I_{ji}$, and $\bar{w} = \frac{1}{ns}\sum_{i=1}^{ns} w_i$. If $x_{1j} < x_{1i}$ or $x_{2j} < x_{2i}$ (x_{1i} and x_{2i} are ith sample point for X_1 and X_2), then $I_{ij} = 1$, otherwise, $I_{ij} = 0$. Thus, the confidence interval for the correlation parameter for the confidence level of $100 \times (1-\alpha)$ is obtained as

$$\Pr\left[\tilde{\theta} - h \le \theta \le \tilde{\theta} + h\right] = 1 - \alpha \quad (23)$$

where $z_{\alpha/2}$ is the CDF value of the Gaussian distribution evaluated at $\alpha/2$ and $h = z_{\alpha/2}\frac{1}{\sqrt{ns}}4w\left|\frac{dg^{-1}(\tilde{\tau})}{d\tilde{\tau}}\right|$.

Using the lower and upper bounds of the confidence interval for the correlation parameter θ, the upper and lower bounds of the confidence interval for the correlation coefficient τ are calculated from $\tau = g(\theta)$ using Eq. (5) or the explicit functions in Ref. 18. To verify Eq. (22), the confidence levels of the 97.5% confidence interval for the correlation coefficient are tested by randomly generating 1000 data sets with $ns = 30$, 100, and 300 from four representative copulas with different correlation coefficient of $\tau = 0.2$, 0.5, and 0.8. The confidence interval for the correlation parameter is accurately estimated regardless of copula function types as shown in Table 5. Thus, the bootstrap method is not necessary to obtain the confidence interval for the correlation parameter in this paper.

Once the input model with the adjusted parameters using the bootstrap method is obtained, the β_t–contour can be obtained as explained in Sec. 2.1. To check how the β_t–contour obtained using the adjusted parameters covers the β_t–contour of the true input model, the confidence level of the input model is measured by counting enough number of data sets that the β_t–contour obtained from the adjusted parameters fully covers the true β_t–contour. The numerical test results are shown in the Sec. 3.5.

3.5 Confidence Levels of Input Model. Let X_1 have a lognormal distribution with $\mu_{X_1} = 5.0$ and $\sigma_{X_1} = 5.0$ (COV=1.0); X_2 have a Gaussian distribution with $\mu_{X_2} = 5.0$ and $\sigma_{X_2} = 1.0$ (COV=0.2). Let X_1 and X_2 are correlated with the Frank copula. From the true input model, a different number of samples, $ns = 30$, 100, and 300, are randomly generated for a sufficient number of trials, 300. Using the generated samples, the marginal distributions and copulas are identified and their parameters are quantified. If the marginal Gaussian distributions are identified, Eq. (10) is used to calculate the confidence interval for the standard deviation. If not, the parametric percentile bootstrap method is used. Then, the input confidence level is assessed by calculating the probability that the obtained β_t–contour covers the true β_t–

Table 5 Confidence level (%) of confidence interval for correlation coefficient using Eq. (22)

τ	ns	Clayton	Gumbel	Frank	Gaussian
0.2	30	95.7	98.7	96.6	94.2
	100	96.9	96.6	97.3	96.8
	300	97.2	97.7	97.3	97.3
0.5	30	95.3	95.6	96.0	99.0
	100	97.2	97.2	96.8	96.6
	300	97.3	97.8	97.6	97.2
0.8	30	96.1	96.1	97.2	92.5
	100	97.3	97.5	96.6	96.2
	300	97.8	97.8	97.1	97.8

Table 6 Input confidence levels (%) using identified CDF types

	$\tau = 0.2$		$\tau = 0.5$		$\tau = 0.8$	
ns	Norm.	Bootstrap	Norm.	Bootstrap	Norm.	Bootstrap
30	83	87	78	84	77	83
100	85	90	85	89	83	87
300	93	96	89	93	90	92

contour over 300 trials. For this test, the target confidence level of 97.5% is used.

Table 6 shows the obtained input confidence levels using the identified marginal distributions and copulas. Since the bootstrap method more accurately calculates the upper bound of the confidence interval for the standard deviation for a non-Gaussian distribution, it yields a more accurate confidence level than the method using the Gaussian distribution for calculation of the upper bound. Likewise, when the true marginal distributions types are used, the input confidence levels using the bootstrap method are more accurate than the method using the Gaussian distribution for calculation of the upper bound as shown in Table 7. When the correct distribution types are used, the performance of the bootstrap method is even more improved as shown in Tables 6 and 7.

To test the confidence levels for various input models, four input models with distinct β_t–contour shapes shown in Fig. 3 are tested. The statistical information on four input models is presented in Table 8.

Randomly generating 100 data sets of $ns = 30$, 100, and 300 from the four input models, the distribution types are identified using the Bayesian method, and the adjusted parameters are estimated using the parametric percentile bootstrap method. As shown in Table 9, even though the input confidence levels for four input models do not reach the target confidence level, 97.5%, they tend to converge to the target as the number of samples increases.

It is possible that the true joint distribution is not one of candidates in the Bayesian method. If so, the obtained input confidence level may not converge to the target confidence level even for infinite number of samples. However, the Bayesian method selects the one that best describes the given data among candidates. Thus, even though the selected distribution may not be the true one, it will yield a similar confidence level as long as the selected distribution has similar distribution shape with the true one.

If the number of samples is very limited, i.e., less than 20, the identification of marginal distributions and copulas could be meaningless. The research on how the confidence levels are incorporated in the problems with very limited data will be a future research.

4 Numerical Examples

In this section, a mathematical example and an M1A1 tank roadarm with correlated non-Gaussian input variables are used to demonstrate how the parametric percentile bootstrap method yields more reliable design than the method assuming a Gaussian distribution of input variable. To carry out RBDO, the MPP-based dimension reduction method (DRM) [34] is used for more accurate calculation of the probability of failure than the first-order reliability method (FORM).

Table 7 Input confidence levels (%) using true CDF types

	$\tau = 0.2$		$\tau = 0.5$		$\tau = 0.8$	
ns	Norm.	Bootstrap	Norm.	Bootstrap	Norm.	Bootstrap
30	86	92	82	90	82	89
100	88	96	90	96	86	92
300	94	97	92	97	92	94

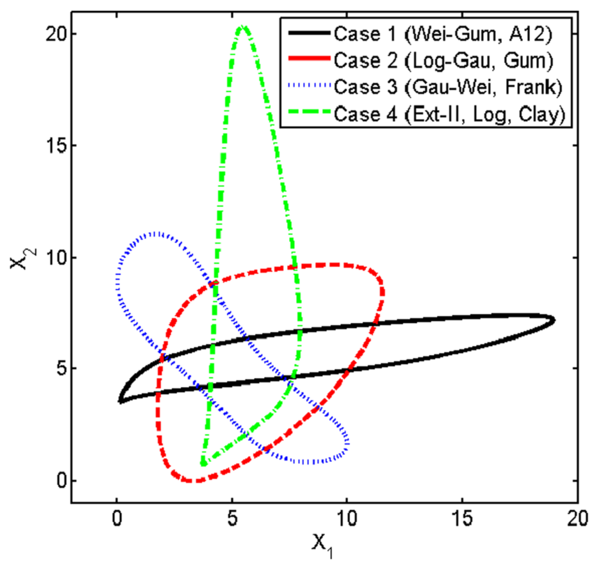


Fig. 3 β_t**–contours for four cases**

4.1 Mathematical Example. Let X_1 and X_2 have lognormal and Gaussian distributions, $X_1 \sim LN(3, 1.5^2)$ and $X_2 \sim N(3, 0.3^2)$, respectively, which are correlated with the Frank copula and $\tau = 0.7$. From the true input model, 100 data sets with $ns = 30$, 100, and 300 are randomly generated, and the marginal distribution, the copula type, and their parameters are determined from each data set.

For the given data, the adjusted standard deviation and correlation coefficients are obtained using Eqs. (7) and (8), respectively. For the adjusted standard deviation for the non-Gaussian distribution, the parametric percentile bootstrap method is used to calculate the confidence interval for the standard deviation. In the RBDO formulation, the input model with the adjusted parameters is used to estimate the probabilistic constraint for RBDO.

For the comparison study, one model with the estimated parameters and another model with the adjusted parameters obtained using the parametric percentile bootstrap method are tested. The two input models are used to estimate the probabilistic constraints of RBDO. The output confidence levels are assessed by counting the number that the probabilities of failure evaluated at optimum designs obtained from 100 data sets are smaller than the target probability of failure, 2.275%.

The RBDO problem is formulated to

$$
\begin{aligned}
&\text{minimize} \quad \text{cost}(\mathbf{d}) = d_1 + d_2 \\
&\text{subject to} \quad P\left(G_j(\mathbf{X}) > 0\right) \le P_{F_j}^{Tar} (= 2.275\%), \ j = 1, 2, 3 \\
&\qquad\qquad \mathbf{d} = \mu(\mathbf{X}), \ 0 \le d_1, d_2 \le 10 \\
&G_1(\mathbf{X}) = 1 - (0.9010X_1 - 0.4339X_2 + 1.5)^2 \\
&\qquad\qquad \times (0.4339X_1 + 0.9010X_2 + 2)/20 \\
&G_2(\mathbf{X}) = 1 - (X_1 + X_2 - 2.8)^2/30 - (X_1 - X_2 - 12)^2/120 \\
&G_3(\mathbf{X}) = 1 - 200/\{2.5(0.9010X_1 - 0.4339X_2 - 3)^2 \\
&\qquad\qquad + 8(0.4339X_1 + 0.9010X_2) + 5\}
\end{aligned}
\tag{24}
$$

where three constraints are shown in Fig. 4.

Table 10 shows the minimum, median, and maximum values of the probabilities of failure P_{F_1} and P_{F_2} for two active constraints G_1 and G_2 evaluated at the optimum designs using the Monte Carlo simulation (MCS), and the output confidence levels of optimum designs, which are obtained from 100 data sets. Figures 5 and 6 show box plots of the statistical test results in Table 10. As

Table 8 Statistical information for four cases

Case	Marginal distribution X_1	Marginal distribution X_2	Copula	Kendall's tau
Case 1	Weibull (COV = 1.0)	Gumbel (COV = 0.2)	A12	0.7
Case 2	Lognormal (COV = 0.5)	Gaussian (COV = 0.5)	Gumbel	0.3
Case 3	Gaussian (COV = 0.5)	Weibull (COV = 0.5)	Frank	−0.7
Case 4	Extreme-II (COV = 0.2)	Lognormal (COV = 1.0)	Clayton	0.5

shown in Table 10, when the input model with the estimated parameters is used for $ns = 30$, 49 and 30 out of 100 data sets for G_1 and G_2, respectively, failed to achieve the target probability of failure. Thus, the confidence levels of the output performance are significantly lower than the target confidence level 97.5% as shown in Table 10. This phenomenon does not change significantly even when the number of samples in each data set increases. As shown in Fig. 5, as the number of sample data increases, boxes whose edges indicate the 25th and 75th percentiles become smaller, which means that the input estimation becomes accurate. However, even with 300 samples, the output confidence level for G_1 is still 52% as shown in Table 10, which is still far less than the target confidence level of 97.5%. In this example, the reason that the confidence level for G_2 is relatively close to the target is because there exists reliability analysis error due to the nonlinearity of G_2 function and highly nonlinear transformation. Even if the MPP-based DRM is used to reduce the reliability analysis error, the probability of failure for G_2 at the optimum design obtained using the true input model is 1.922%, which is less than the target. Thus, as the number of samples increases, probabilities of failure at optimum designs obtained using the estimated parameters tend to converge to 1.922% instead of 2.275%.

On the other hand, when the input model with the adjusted parameters obtained using the parametric percentile bootstrap method is used, the median values of P_{F_1} and P_{F_2} are smaller than 2.275%, which yields more confidence than the case of the estimated parameter. Accordingly, the obtained output confidence levels using the adjusted parameters is much closer to the target confidence level, 97.5% as shown in Table 6. As the number of samples increases, the output confidence levels using the parametric percentile bootstrap method are getting much closer to the target confidence level as shown in Fig. 6, whereas the input model with the estimated parameters does not provide the output confidence level near the target confidence level as shown in Table 5. Also, as the number of samples increases, the average cost value at the optimum design decreases, which means that we can obtain a better optimum design if we can spend more on accurate input estimation.

4.2 M1A1 Abrams Tank Roadarm. The roadarm in the M1A1 tank is modeled using 1572 eight-node isoparametric finite elements (SOLID45) and four beam elements (BEAM44) of a commercial program, ANSYS [35], as shown in Fig. 7. The material of the roadarm is S4340 steel with Young's modulus $E = 3.0 \times 10^7$ psi and Poisson's ratio $\nu = 0.3$. The durability analysis of the roadarm is carried out to obtain the fatigue life contour using the Durability and Reliability Analysis Workspace (DRAW) code [36,37]. The fatigue lives at the 13 critical nodes are selected for the design constraints of the RBDO. Detailed information on the location of 13 critical nodes and design parameterization of the roadarm can be found in Ref. 34.

Table 9 Input confidence level using parametric percentile bootstrap method

Ns	Case 1	Case 2	Case 3	Case 4
30	88	85	82	88
100	96	92	92	94
300	98	97	98	98

Table 11 shows the assumed statistical information of random variables and parameters for RBDO of the roadarm. Since the true statistical information on S4340 steel except its nominal values is not available, it is assumed in the paper using the study of SAE 950X.

First, it is assumed that Frank copula ($\tau = -0.683$) for σ'_f and b, and Gaussian copula ($\tau = -0.906$) for ε'_f and c, respectively, are the true copulas. As the two copulas well describe the experimental data of SAE 950X [19] as shown in Fig. 8, it seems to be reasonable to select these two copulas to model the joint CDFs of the four correlated random parameters of S4340 steel. Furthermore, the marginal distribution types of S4340 steel are assumed to be the same as those of SAE 950X.

Second, once the copula and marginal distribution types are obtained, the mean and standard deviation of S4340 need to be determined. The mean values of four fatigue material properties of S4340 are obtained from a materials standard book, but the standard deviations are unknown. Therefore, the standard deviations are assumed using COV of SAE 950X. The coefficient of variation of SAE 950X is 115% for ε'_f and 25% for other material properties [19]. Since S4340 steel is a stronger material than SAE 950X, in this paper, it is assumed that the COV of S4340 is 50% for ε'_f and 25% for other material properties to estimate the standard deviation as shown in Table 11.

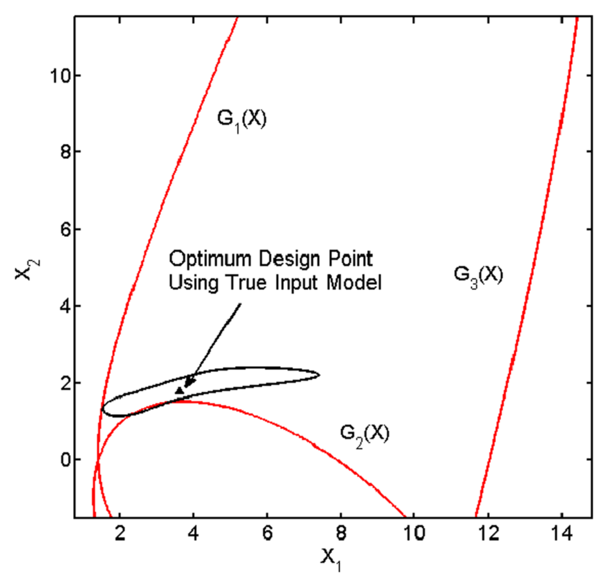


Fig. 4 Optimum design point using true input model for Eq. (24)

Table 10 Probabilities of failure and output confidence levels

RBDO with	*ns*	P_{F_1},% Minimum	Median	Maximum	CL[a]	P_{F_2},% Minimum	Median	Maximum	CL	Average cost at optimum
Estimated parameter	30	0.128	2.149	22.003	**51**	0.158	1.515	11.982	**70**	5.376
	100	0.369	2.329	8.759	**48**	0.262	1.556	5.010	**84**	5.422
	300	0.805	2.246	4.689	**52**	0.842	1.709	3.081	**90**	5.434
Adjusted parameter (Percentile)	30	0.000	0.273	4.162	**93**	0.004	0.466	8.082	**92**	6.374
	100	0.003	0.551	4.687	**95**	0.122	1.018	4.788	**93**	6.032
	300	0.329	0.954	2.847	**96**	0.582	1.540	2.611	**97**	5.792

[a]Indicates the confidence level.

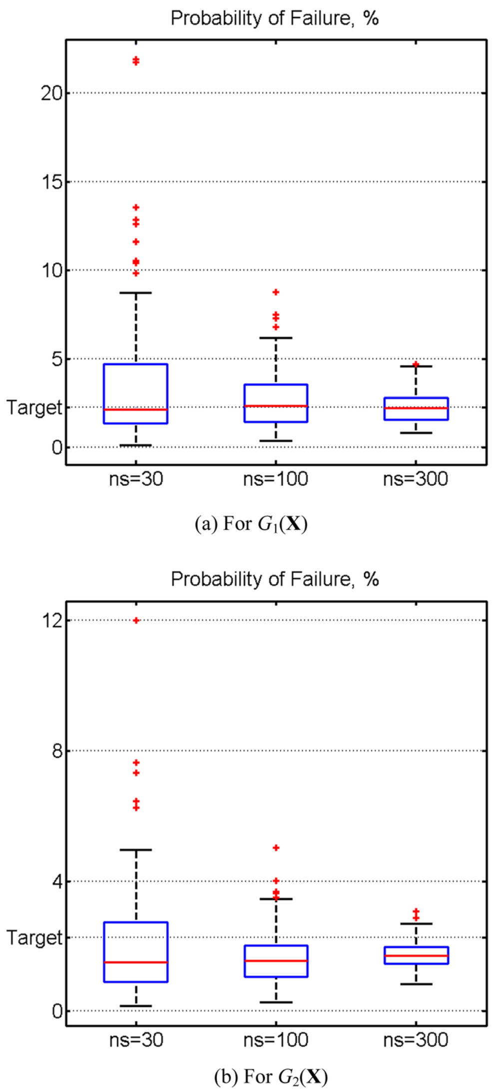


Fig. 5 Box plot for estimated parameters

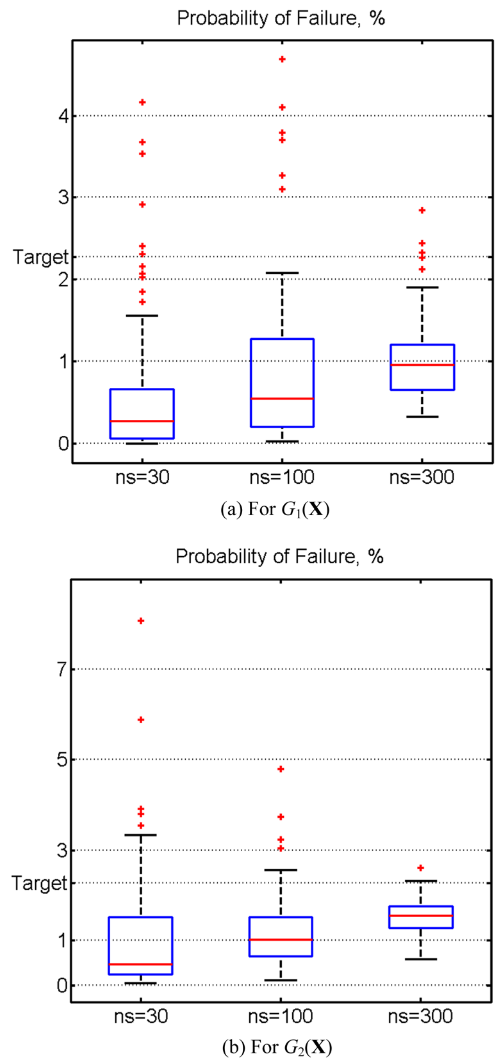


Fig. 6 Box plot for adjusted parameters

Assuming that S4340 steel has the true statistical information as shown in Table 11, 30 paired data are randomly generated from the assumed true statistical information, and two input statistical models with estimated and adjusted parameters, respectively, are computed using the generated 30 data and will be used to carry out RBDO. Table 12 shows the estimated and adjusted parameters, and the target confidence level is specified as 97.5% in this roadarm example.

The RBDO for the roadarm is formulated to

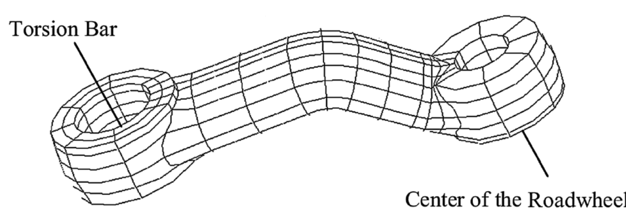


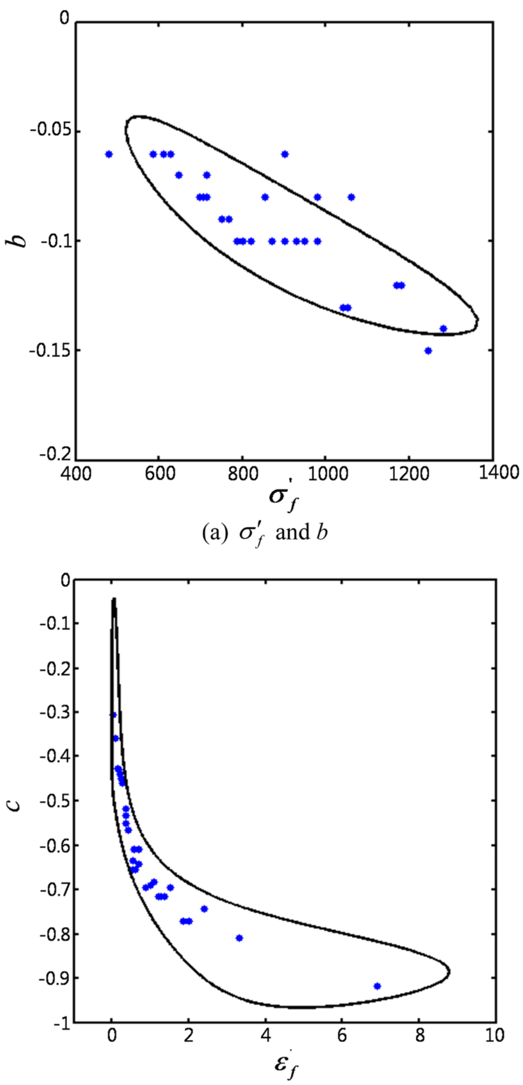
Fig. 7 Finite element model of roadarm

Table 11 Random variables and fatigue material properties

Random variables	Lower bound $\mathbf{d}^L$	Initial design $\mathbf{d}^0$	Upper bound $\mathbf{d}^U$	Standard deviation	Distribution Type
d_1	1.3500	1.7500	2.1500	0.0875	Gaussian
d_2	2.6496	3.2496	3.7496	0.1625	Gaussian
d_3	1.3500	1.7500	2.1500	0.0875	Gaussian
d_4	2.5703	3.1703	3.6703	0.1585	Gaussian
d_5	1.3563	1.7563	2.1563	0.0878	Gaussian
d_6	2.4377	3.0377	3.5377	0.1519	Gaussian
d_7	1.3517	1.7517	2.1517	0.0876	Gaussian
d_8	2.5085	2.9085	3.4085	0.1454	Gaussian

Fatigue material properties

Nondesign uncertainties	Mean	Standard Deviation	DistributionType
Fatigue strength coefficient, σ_f'	177000	44250	Lognormal
Fatigue strength exponent, b	−0.0730	0.018	Gaussian
Fatigue ductility coefficient, ε_f'	0.4100	0.205	Lognormal
Fatigue ductility exponent, c	−0.6000	0.150	Gaussian



(a) σ_f' and b

(b) ε_f' and c

Fig. 8 Joint PDF contours of Gaussian and Frank copula identified from 29 paired data of SAE 950X

$$
\begin{aligned}
&\text{minimize} \quad \text{cost}(\mathbf{d}) \\
&\text{subject to} \quad P(G_j(\mathbf{X}) \geq 0) \leq P_{F_j}^{Tar}, \; j = 1 \sim 13 \\
&\mathbf{d} = \mu(\mathbf{X}), \; \mathbf{d}^{\mathbf{L}} \leq \mathbf{d} \leq \mathbf{d}^{\mathbf{U}}, \; \mathbf{d} \in R^n, P_{F_j}^{\mathrm{Tar}} = 2.275\% \\
&G_j(\mathbf{X}) = 1 - \frac{L(\mathbf{X})}{L_t}, \; j = 1 \sim 13 \\
&\text{cost}(\mathbf{d}): \; \text{Weight of Roadarm} \\
&L(\mathbf{X}): \; \text{Crack Initiation Fatigue Life,} \\
&L_t: \; \text{Crac Initiation Target Fatigue Life} (= 5\text{years})
\end{aligned} \tag{25}
$$

Table 13 shows a comparison of RBDO results for various input models where the assumed true input model is the one in Table 11. First, the RBDO results are compared for the independent and correlated input fatigue material properties. As shown in the table, when the correlation between material properties is considered, the optimized weight of the roadarm is significantly reduced from 592.22 to 514.02 for the same target reliability. This is because the material properties are highly and negatively correlated as shown in Fig. 8. Thus, it is very important to correctly model the correlation between material properties to carry out the RBDO.

Second, when the estimated input model is used, the underestimated standard deviations (see Tables 11 and 12) yield a smaller optimum cost than the optimum cost obtained using the true input model (509.44 versus 514.02). On the other hand, when the input model with the adjusted parameters is used, the obtained optimum cost is higher than the optimum cost obtained from the true input model (531.64 versus 514.02). Since the MCS cannot be used for the benchmark test for this problem due to computational cost, it is difficult to determine which input model yields reliable optimum designs by comparing the optimum costs. Thus, at each optimum design, fatigue analysis is carried out using the true input

Table 12 Estimated and adjusted parameters

	σ_f'	b	ε_f'	c
$\tilde{\mu}$	176738	–0.073	0.395	–0.594
$\tilde{\sigma}$	35141	0.015	0.143	0.110
$\tilde{\sigma}^A$	45356	0.020	0.223	0.155
$\tilde{\tau}$	−0.701		−0.921	
$\tilde{\tau}^A$	−0.596		−0.866	
Copula	Frank		Gaussian	

Table 13 Comparison of RBDO results

	Initial	Independent	Correlated		
			True	Estimated	Adjusted
d_1	1.750	2.194	1.928	1.954	2.052
d_2	3.250	2.650	2.650	2.650	2.650
d_3	1.750	2.602	2.067	2.030	2.038
d_4	3.170	3.010	2.577	2.623	2.702
d_5	1.756	2.656	1.776	1.684	1.804
d_6	3.038	2.538	3.535	3.538	3.500
d_7	1.752	2.422	2.075	2.025	2.152
d_8	2.908	2.895	2.512	2.508	2.754
Cost	515.09	592.22	514.02	509.44	531.64

model in Table 11. At the optimum design obtained using the estimated input model, the lowest fatigue life is 1.5 years, which occurs at the MPP of G_{13}. This fatigue life is much less than the target (5 years). On the other hand, at the optimum design obtained using the adjusted parameters, the lowest fatigue life is 6.16 years, which occurs at the MPP of G_4, which is larger than the target fatigue life. Accordingly, the input model with the adjusted standard deviation and correlation coefficient is indeed necessary to obtain a reliable optimum design.

5 Conclusions

In many engineering applications, only limited test data are available for input variables, and thus, the input statistical model obtained from the insufficient data could yield an unreliable optimum design. Thus, the RBDO with confidence level is proposed to offset the inaccurate estimation of the input model by using the adjusted standard deviation and correlation coefficient. The adjusted standard deviation is obtained from the confidence intervals for the standard deviation, mean, and correlation coefficient. If the input variables have a Gaussian distribution, the method using Gaussian distribution of input variables is explicit and exact. If not, it yields an inaccurate estimation of the confidence interval for the standard deviation. Thus, in this paper, the bootstrap method is used to calculate the confidence interval for the standard deviation, and thus, the adjusted standard deviation. The input model with the adjusted parameters obtained from the bootstrap method is used to assess the input and output confidence levels for the non-Gaussian distributions. Numerical test shows that the percentile method has the most desirable performance out of five candidate bootstrap methods for the parametric approach. Numerical examples also show that the input model using the parametric percentile bootstrap method yields more reliable design than the one using other methods for the non-Gaussian distributions.

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